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Some Numerical Approaches to Solving One-Dimensional Inverse Problems*

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A class of one-dimensional inverse scattering problems are studied with the goal of reconstructing (say) propagation speed to moderate accuracy as inexpensively as possible. Three alternatives are discussed all starting from a change to the "travel-time" variable and converting the problem to integral equation form. The approaches are compared with regard to their economy of use and the problems for which they are effective. Several numerical examples illustrate these comparisons.

I. INTRODUCTION

We will be considering one-dimensional inverse problems of the form

$$\begin{aligned} &y''(x) + \frac{\omega^2}{v(x)^2}\, y(x) = 0, \\ &y'(0) + \frac{i\omega}{v(0)}\, y(0) = 2i\omega, \\ &y'(X) - \frac{i\omega}{v(X)}\, y(X) = 0, \end{aligned} \tag{1'}$$

where $X \leqslant \infty$. The goal is to reconstruct the unknown (say) velocity v from certain data prescribed at a set of ω values. The real parameter ω arises from an earlier Fourier transform of the original wave equation.

Similar problems have been studied by Gray [1, 2], Hagin [2, 3], Chen and Tsien [4] and others. Our purpose here is to investigate the computational feasibility of some of these ideas and, in addition, to introduce another alternative (Section III) that in some cases is very attractive.

In [1] the "travel time" variable change was shown to be effective in certain

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instances. Our investigations have confirmed that this is in general a good starting point; so we define

$$\tau = \int_0^x ds/v(s)$$

as our new independent variable and let $u(\tau) = y(x)$ and $c(\tau) = v(x)$. The boundary value problem now becomes

$$\begin{aligned} u'' - c'/c\, u' + \omega^2 u(\tau) &= 0, \\ u'(0) + i\omega u(0) &= c(0)\, 2i\omega, \\ u'(T) - i\omega u(T) &= 0, \end{aligned} \tag{1}$$

where $T = \tau(X) \leqslant \infty$. Since we will soon truncate (if necessary) and discretize for numerical computations, let us assume that $T < \infty$. For convenience we will assume that both $T = 1$ and $c(0) = 1$ (which could be accomplished by scaling; see the discussion at the end of this section). Finally, it will be convenient to have the differential equation in self-adjoint form, and this leads to the statement

$$\begin{aligned} (u'/c)' + \omega^2 u/c &= Lu = 0, \\ [u' + i\omega u](0) = 2i\omega, \qquad & [u' - i\omega u](1) = 0. \end{aligned} \tag{2}$$

A central notion is that of an "incident wave," denoted herein by u_i. For example, our problem could be interpreted as the result of wave $u_i = e^{i\omega\tau}$ moving in from the region $\tau \leqslant 0$ in which the medium has constant propagation velocity $c = 1$. More generally, from our viewpoint u_i will denote some first approximation to the solution $u(\tau; \omega)$ of (2) in which u_i does satisfy the boundary conditions.

To obtain an integral equation for the unknown c we proceed as follows. Applying the Lagrange identity to (2), using the fact $Lu = 0$, and defining $u_s = u - u_i$ we obtain

$$\begin{aligned} \int_0^1 (uLu_i - u_i Lu) = \int_0^1 uLu_i &= \int_0^1 (u_i + u_s)\, Lu_i \\ &= [uu_i' - u'u_i]_0^1 = -2i\omega u_s(0; \omega). \end{aligned} \tag{3}$$

The values $u_s(0; \omega)$ for some ω set are our data and will be assumed known for the ω values of our choice. This integral equation for c is still not convenient since it involves the unknown u or u_s (although in Section IV we will attack it via iteration). The problem is much more manageable if we are justified in assuming that u_s is small in comparison to u_i so that the u_s part of (3) can be dropped to obtain

$$\int_0^1 u_i Lu_i = -2i\omega u_s(0; \omega). \tag{4}$$

In (4), u_i will be a known estimate and Lu_i will involve the unknown c; thus we obtain a linear integral equation for c. In the next two sections we turn to two natural selections for u_i; $e^{i\omega\tau}$ and $\sqrt{c}\,e^{i\omega\tau}$. We shall see that if c'/c is relatively small, the first of these leads to an exceptionally inexpensive method for reconstructing c (as in [1]). If c'/c is not small enough then $u_i = e^{i\omega\tau}$ is not an adequate estimate for u and hence the neglected u_s term in (3) is significant. In Section III we show that in this case, and if c is sufficiently well-behaved, the WKB approximate $\sqrt{c}\,e^{i\omega\tau}$ is a practical choice which leads to a fairly inexpensive algorithm for reconstructing c. Finally, in Section IV we turn to the more general situation in which we return to (3) and through iteration involve the u_s term neglected in (4). Although this process is considerably more expensive than the earlier alternatives, it is certainly not prohibitive; moreover, it is probably unavoidable in many problems.

Before proceeding to our first approach to reconstructing c, we comment on the practicality of assuming in (1) that $T = \tau(X)$ and $c(0)$ can be normalized to one. In many applications $\tau = 0$, or $x = 0$, represents the "surface" of observation, in which case $c(0)$ is known. It is then a simple matter to define, say, $\bar{c} = c/c(0)$ in (1) to make $\bar{c}(0) = 1$.

The $T = 1$ scaling is a bit more subtle. Note that the boundary condition at $\tau = T$ in (1), or $x = X$ in (1′), can be viewed as assuming that the wave is "right going," i.e., $u(\tau) = Ae^{i\omega\tau}$ for $\tau > T$. Physically this suggests that no reflections occur from the region $\tau > T$, i.e., $c(\tau) = c(T)$ for $\tau > T$. An alternate explanation is: if T is sufficiently large then reflections from the $\tau > T$ region simply do not get back to the $\tau = 0$ surface in time to be recorded in the data. Whatever the explanation for the right boundary condition, T could be replaced by any $T' > T$ without affecting the solution of (1). This observation plus the fact that one normally has a rough estimate of $c(\tau)$ means that one can at least estimate T. Hence we view T as known. It can then be normalized to one in (1) by letting $z = \tau/T$; also let $U(z) = u(\tau)$, $C(z) = c(\tau)$, and $w = T\omega$. It is easily verified that the resulting differential equation in $U(z)$ is precisely the same as that in (1), but now takes place for $0 \leqslant z \leqslant 1$. Hence, at least in many cases, we are justified in assuming that $c(0) = 1 = T$.

II. Results Using $u_i = e^{i\omega\tau}$

For this choice for u_i, it will be convenient to introduce the operator L_0 given by $L_0 v = v'' + \omega^2 v$. Note that $L_0 u_i = 0$. Proceeding as in the derivation of (3), we obtain

$$\int_0^1 [u_i L_0 u - u L_0 u_i] = \int_0^1 u_i L_0 u = \int_0^1 e^{i\omega\tau}(c'/c)\,u'$$

$$= \int_0^1 e^{i\omega\tau}[i\omega e^{i\omega\tau} + u_s'](c'/c)\,d\tau = 2i\omega u_s(0;\omega). \tag{5}$$

In [2] it was shown that if $\gamma \equiv c'/c = O(\varepsilon) < 1$, in the L^1 norm, then the term in (5)

involving u_s' is $O(\varepsilon^3)$. Hence if γ is indeed small, one is justified in dropping that term to get

$$\int_0^1 e^{2i\omega\tau}\gamma(\tau)\,d\tau \approx 2u_s(0;\omega). \tag{6}$$

Cearly (6) can be solved easily for $\gamma = c'/c$ via Fourier methods assuming data $u_s(0;\omega)$ are available for the required ω values. Suppose, for simplicity of presentation, that we have reason to believe that $\gamma(0)$ and $\gamma(1)$ are "small," so that it is reasonable to represent γ by its Fourier sine series. (If not, one could use the cosine series or the full series and proceed accordingly.) By taking the imaginary part of (6) and setting $\omega = \omega_k = k\pi/2$ we have

$$\int_0^1 \sin k\pi\tau\ \gamma(\tau)\,d\tau = 2\ \mathrm{Im}\ u_s(0;\omega_k) = b_k/2, \tag{7}$$

where b_k are the (approximate) Fourier coefficients of γ. Finally we have

$$\gamma(\tau) = \sum_1^\infty b_k \sin k\pi\tau \approx \sum_1^N b_k \sin k\pi\tau$$
$$= 4\sum_1^N \mathrm{Im}\ u_s(0;\omega_k)\sin k\pi\tau.$$

Since $\gamma = c'/c$ we have, from integration,

$$c(\tau) = \exp\left[4\sum_1^N \mathrm{Im}\ u_s(0;\omega_k)(1-\cos k\pi\tau)/k\pi\right]. \tag{8}$$

(Formula (8) amounts to a discrete version of the approach of [1] in which the $T = \infty$ case was treated using Fourier transform.)

In [2] it was shown that if (6) is solved for γ the error due to ignoring the u_s term in (5) is approximately $d^3/4\pi$, where d is the L^1 norm of γ. For example, if we are seeking two-figure accuracy, this suggests that the relative error (in L^1 norm) should be less than 0.01; i.e., $d^2/4\pi \approx d^2/10 < 0.01$ or

$$d = \|\gamma\| < 0.3. \tag{9}$$

At least (9) should serve as a rule of thumb for how large $\gamma = c'/c$ can be for (8) to produce two figure results. In the following example we investigate the situation numerically.

EXAMPLE 1. Consider the function

$$c(\tau) = 1 + \varepsilon \sin^2(\pi\tau/2) \qquad \text{for } \tau \text{ in } [0, 1].$$

TABLE I

Errors in Solving for
$c = 1 + \varepsilon \sin^2(\pi\tau/2)$ via (8)[a]

ε	Max error	Mean error
0.3	0.001	0.0003
0.7	0.011	0.0039
0.8	0.016	0.0057
1.0	0.030	0.0106
2.0	0.194	0.0715

[a] Time: 0.17 sec per ε.

Since $\gamma = c'/c \approx c' = \varepsilon\pi/2 \sin \pi\tau \approx \varepsilon$, (9) suggests that we should be able to achieve two-figure results with ε roughly as large as 0.3. The data $u_s(0; \omega_k)$ were created artificially for this and subsequent examples by using the exact c in (2) to generate $u(\tau; \omega_k)$ and then $u_s = u - u_i$ numerically. Then c is reconstructed by using, in this case, (8) above with $N = 10$. Table I summarizes the numerical results. For several values of ε ranging between 0.3 and 2.0 two types of errors are listed: the "maximum error" over the 10 computed points and the "mean error" (which is the discrete version of the L^1 error). Note that we achieved the desired two-figure accuracy for $\varepsilon < 0.8$ and lose it for $\varepsilon > 0.8$; hence the indication given by (9) above is somewhat conservative in this instance. The results are illustrated graphically in Fig. 1.

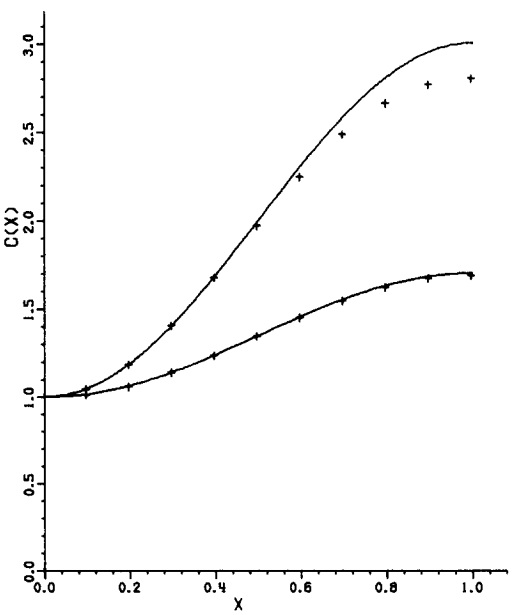

FIG. 1. Graphs of $c = 1 + \varepsilon \sin^2(\pi\tau/2)$ for $\varepsilon = 0.7$ and 2, and their approximations due to (8) denoted by +.

In the next section we refine the procedure just discussed in order to handle some cases in which $\gamma = c'/c$ is not "small."

III. Results Using $u_i = \sqrt{c}\, e^{i\omega\tau}$

One way of viewing the results of the last section is: if c varies little (c'/c small) then $u_i = e^{i\omega\tau}$ is an effective estimate for $u(\tau; \omega)$; hence dropping the $u_s = u - u_i$ term in (3) does little harm. We now look at a procedure which is effective in at least some cases in which c'/c is not small. We choose as u_i the WKB approximation

$$u_i = \sqrt{c}\, e^{i\omega\tau}.$$

As interpreted physically in Bremmer [5], if a wave $e^{i\omega\tau}$ for $\tau < 0$ enters the medium at $\tau = 0$, then $\sqrt{c}\, e^{i\omega\tau}$ is the approximation to u obtained by considering only the primary reflections and transmissions due to the inhomogeneous medium (c non-constant). From a mathematical point of view, $\sqrt{c}\, e^{i\omega\tau}$ represents the first two terms in a large ω expansion of solution u, whereas $e^{i\omega\tau}$ is the first term (e.g., see [6]). This comparison can be illustrated as follows. Consider the operator L of (2); recall that we seek a solution to $Lu = 0$. It is easy to show that

$$\begin{aligned} L(e^{i\omega\tau}) &= -i\omega\gamma/c\, e^{i\omega\tau}, \\ L(\sqrt{c}\, e^{i\omega\tau}) &= 1/2\, c^{-1/2}(c''/c - 3/2\, (c'/c)^2)\, e^{i\omega\tau}. \end{aligned} \tag{10}$$

Viewing the right sides in (10) as "error," we see that $L(e^{i\omega\tau}) = O(\omega)$ while $L(\sqrt{c}\, e^{i\omega\tau}) = O(1)$ in ω. In practice $\sqrt{c}\, e^{i\omega\tau}$ is an excellent approximation to u for large ω and is often a surprisingly good estimate for moderate and even "small" ω. Equally important in the right sides of (10) are the expressions involving c. In the first of these note that $\gamma = c'/c$ appears, supporting the "small" γ assumption of the last section. In the second equation γ^2 appears, as does a c'' term. In particular the c'' forewarns us that c may have to be quite smooth and that c'' cannot be too large. This is borne out in the examples below.

To obtain an integral equation we use (4) with $u_i = \sqrt{c}\, e^{i\omega\tau}$ and (10),

$$\int_0^1 u_i L u_i = 1/2 \int_0^1 e^{2i\omega\tau}(c''/c - 3/2\, \gamma^2)\, d\tau = -2i\omega u_s(0; \omega).$$

Analogous to the approach of Section II, we choose $\omega = \omega_k = k\pi/2$ for $k = 1, 2, \ldots$ and take the real part to obtain

$$\int_0^1 \cos k\pi\tau(c''/c - 3/2\, \gamma^2)\, d\tau = 2k\pi \operatorname{Im} u_s(0; \omega_k) = 1/2\, a_k, \tag{11}$$

where the a_k are the Fourier coefficients in

$$c''/c - 3/2\,(c'/c)^2 = \sum_0^\infty a_k \cos k\pi\tau = S_\infty(\tau). \tag{12}$$

Perhaps the most straightforward way to proceed is to truncate the series in (12) to give a trigonometric polynomial $S_N(\tau)$ and solve the resulting second-order differential equation numerically for c. In practice, however, this does not work unless c is exceptionally smooth. Note that the series in (12) is for a function involving c''; hence unless c is very smooth the series will converge very slowly, if at all. Ideally one would like to integrate the series in (12) once or twice term-by-term in order to improve its behavior. We now move in that direction.

Observe that

$$c''/c - 3/2\,(c'/c)^2 = (c'/c)' - 1/2\,(c'/c)^2 = \gamma' - 1/2\,\gamma^2.$$

Using this in (11) we can write

$$\begin{aligned}\int_0^1 \gamma' \cos k\pi\tau\, d\tau &= 1/2\left[a_k + \int_0^1 \gamma^2 \cos k\pi\tau\, d\tau\right] \\ &= 1/2\,A_k \qquad \text{for} \quad k = 1, 2,...,\end{aligned} \tag{13}$$

where $a_k = 4k\pi \operatorname{Im} u_s(0; \omega_k)$ as in (11) and A_k are the coefficients in

$$\gamma' = \sum_1^\infty A_k \cos k\pi\tau. \tag{14}$$

The coefficient $A_0 = \int \gamma' = \gamma(1) - \gamma(0)$ requires special consideration, so to simplify the presentation we will, as we did in Section II, assume that $c'(0) = c'(1) = 0$ and hence that $A_0 = 0$. We now define our procedure for iteratively computing the A_k and ultimately c itself. We first compute γ_0' by ignoring the a_k correction in (13) and proceed as follows:

$$\begin{aligned}\gamma_0'(\tau) &= \sum_1^N a_k \cos k\pi\tau, \\ \gamma_0(\tau) &= \sum_1^N a_k/k\pi \sin k\pi\tau, \\ c_0(\tau) &= \exp\left[\sum_1^N a_k/(k\pi)^2(1 - \cos k\pi\tau)\right].\end{aligned} \tag{15}$$

A comparison with the results of Section II shows that c_0 is precisely that c computed by the $u_i = e^{i\omega\tau}$ assumption. Using this as a starting point we iterate as follows:

1. Using current estimate for γ in the right side of (13), compute the A_k for $k = 1, 2, \ldots, N$.

2. Using the updated A_k, compute γ', γ, and c by (15) with A_k replacing the a_k.

Steps 1 and 2 are repeated as necessary to achieve convergence of c to the desired accuracy. For example, when seeking two figure results we typically repeated these steps twice. Although a rigorous proof of convergence would no doubt require that γ^2 be "small" compared to γ', we have had success when these two quantities were about the same size.

We now consider two examples in which c will be reconstructed by the algorithm just described. As before, the data $u_s(0; \omega_k)$ were constructed artificially by using the known c in (2). In the first of these two examples we again use the c of Example 1 above for comparison.

EXAMPLE 2. Consider the c of Example 1,

$$c(\tau) = 1 + \varepsilon \sin^2(\pi\tau/2).$$

Recall that in Section II, we were able to reconstruct c only to one figure (about 10% error) as ε approached 2. Table II compares the results of Section II (producing c_0) to the results of two corrections (iterations) as outlined above (producing c_2). In particular, we are able to achieve two-figure accuracy for ε up to 2. In this example and the next we only corrected the first four a_k in (13) as an economy measure (clearly for a reasonably smooth function the first several Fourier coefficients are the most significant). The computing times on a Burroughs 6800 was 0.17 sec for c_0 and 1.72 sec for c_2.

A second set of figures are included in Table II under the heading "undercorrecting in (13)" for the following reason. In our numerical work we have observed that in "correcting" the coefficients via (13), there has been a tendency to overcorrect. The actual coefficients are consistently somewhere between a_k and A_k, typically about $0.2a_k + 0.8A_k$. For that reason we can improve our results (plus save computing time) by putting a factor of, say, 0.8 in front of the correcting integral in (13) and

TABLE II

Errors in Solving for $c = 1 + \varepsilon \sin^2(\pi\tau/2)$

	Two corrections via (13)–(15)[a]		Undercorrecting in (13)[b]	
ε	Max error	Mean error	Max error	Mean error
0.8	0.009	0.0045	0.006	0.0030
1.0	0.013	0.0061	0.008	0.0037
2.0	0.025	0.0143	0.012	0.0063

[a] Time: 1.72 sec per ε.

[b] Time: 0.98 sec per ε.

TABLE III

Errors in Solving for the Ramp Function (16)

c_0 via (15)[a]		Correcting via (13)–(15)[b]		Undercorr. in (13)[c]	
Max err	Mean err	Max err	Mean err	Max err	Mean err
0.129	0.051	0.038	0.019	0.031	0.014

[a] Time: 0.17 sec.
[b] Time: 3.87 sec.
[c] Time: 2.00 sec.

only making one iteration. Since this phenomenon is not fully understood, we only make this passing remark and illustrate the numerical results. (It seems to suggest that for some reason a weighted average of $e^{i\omega\tau}$ and $\sqrt{c}\,e^{i\omega\tau}$ is a superior u_i for these purposes.)

Recall that in this development the factor c'' plays a central role and, as pointed earlier, this forewarns one that c'' will have to remain relatively small. In spite of this, we are able to handle the following c which is continuous but c' is discontinuous; hence technically c'' is a delta function.

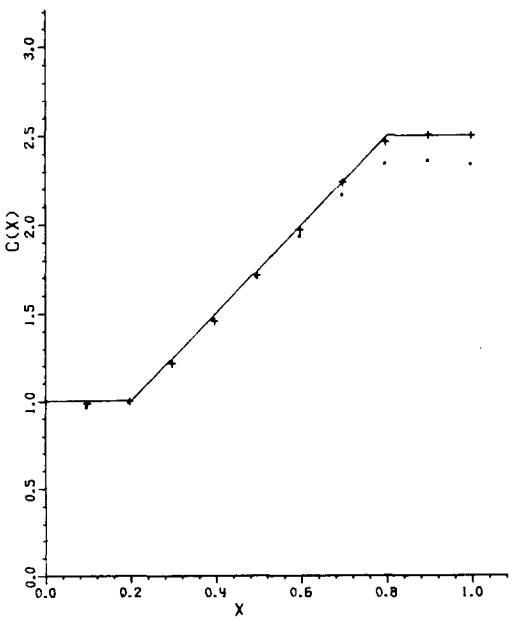

FIG. 2. Graph of the ramp function (16) and its approximation due to two corrections by (13)–(15) denoted by +; and by (15) denoted by ·.

EXAMPLE 3. Consider the "ramp" function:

$$\begin{aligned} c(\tau) &= 1 && \text{for} \quad 0 \leqslant \tau < 0.2 \\ &= 1 + \tfrac{5}{2}(\tau - 0.2) && \text{for} \quad 0.2 \leqslant \tau < 0.8 \\ &= 2.5 && \text{for} \quad 0.8 \leqslant \tau \leqslant 1.0. \end{aligned} \tag{16}$$

In spite of the "corners" in this function, the first 10 terms of the Fourier expansion are again adequate for our purposes. Here we are able to ahieve almost two-figure results with two corrections (or with undercorrecting once). The results are summarized in Table III and shown graphically in Fig. 2.

The above example pushes the algorithm of this section to its limit. For example, if c were discontinuous or if it were smooth but such that c'' is large (say, greater than 10) the numerical results typically would be little better than those obtained by the simpler method of Section II. We shall give examples of this in the next section. Moreover, if more than two-figure accuracy is needed the approaches of the last two sections are usually not adequate. In such cases there seems to be little alternative to some form of iteration. We now turn to a straightforward iteration scheme.

IV. ITERATION

In preparation for iteration we return to (5) and put the u_s' term on the right:

$$i\omega \int_0^1 e^{2i\omega\tau}\gamma \, d\tau = 2i\omega u_s(0;\omega) - \int_0^1 \gamma u_s'(\tau;\omega)\, e^{i\omega\tau}\, d\tau.$$

Once again we will assume, for simplicity, that $c'(0) = c'(1) = 0$ so that the sine series of γ is appropriate. Hence in the above equality if we divide out the $i\omega$, take the imaginary part, and set $\omega = \omega_k = k\pi/2$ we get

$$\begin{aligned} \int_0^1 \gamma \sin k\pi\tau \, d\tau &= 2 \operatorname{Im} u_s(0;\omega_k) + \frac{1}{\omega_k} \operatorname{Re} \int_0^1 \gamma u_s'(\tau;\omega_k)\, e^{ik\pi\tau/2}\, d\tau \\ &= 1/2\, b_k + \frac{1}{\omega_k} \operatorname{Re} \int_0^1 \gamma u_s'(\tau;\omega_k)\, e^{ik\pi\tau/2}\, d\tau \\ &= 1/2\, B_k; \qquad k = 1, 2, \ldots. \end{aligned} \tag{17}$$

In (17) the b_k are the approximate Fourier coefficients of Section II and the B_k represent the exact coefficients in

$$\gamma(\tau) = \sum^{\infty} B_k \sin k\pi\tau.$$

Of course the B_k are not directly available since the integral in the right side of (17) involves the unknown γ (both explicitly and implicitly through $u_s' = u_s'(\tau; \omega_k; c)$); hence it will be necessary to iterate. We use the γ and c of Section II as our starting values; i.e., first one computes

$$\gamma_0(\tau) = \sum_1^N b_k \sin k\pi\tau = 4 \sum_1^N \operatorname{Im} u_s(0; \omega_k) \sin k\pi\tau,$$
$$c_0(\tau) = \exp\left[\frac{1}{\pi} \sum_1^N b_k(1 - \cos k\pi\tau)/k\right]. \tag{18}$$

One can then use (17) for iteration, particularly if c is very smooth. However, generally it is better to remove the $\gamma = c'/c$ from the computation in the right side by integration by parts. Writing the integrand as $c'[(u_s'/c)\, e^{i\omega\tau}]$ and integrating c' and differentiating the second term leads, after considerable simplification, to this replacement of (17):

$$\int_0^1 \gamma \sin k\pi\tau \, d\tau = 1/2\, B_k = -k\pi \int_0^1 \ln c \cos k\pi\tau \, d\tau$$
$$+ [(\operatorname{Im} u + \operatorname{Re} u'/\omega_k) \cos \omega_k$$
$$+ (\operatorname{Re} u - \operatorname{Im} u'/\omega_k) \sin \omega_k]_{\tau=1}. \tag{19}$$

The iteration procedure is therefore:

1. Using most recent estimate for c and with $\omega_k = k\pi/2$ for $k = 1, 2, \ldots, N$, numerically solve, for $\tau \leqslant 1$, the initial value problems

$$(u'/c)' + (\omega_k^2/c)u = 0,$$
$$u(0; \omega_k) = u_s(0; \omega_k) + 1, \tag{20}$$
$$u'(0; \omega_k) = u_s'(0; \omega_k) + i\omega_k = i\omega_k[1 - u_s(0; \omega_k)].$$

Compute B_k using this c and the computed values of u and u' (at $\tau = 1$) in the right side of (19).

2. Compute a new estimate of c and γ by

$$\gamma(\tau) = \sum_1^N B_k \sin k\pi\tau,$$
$$c(\tau) = \exp\left[1/\pi \sum_1^N B_k(1 - \cos k\pi\tau)/k\right]. \tag{21}$$

Note in (19) the data $u_s(0; \omega_k)$ no longer appear explicitly; however, they are involved in the values of u and u' at $\tau = 1$ since the data define the initial values of u in (20). We have recommended the solution of the initial value problem (20) rather

than the boundary value problem (2) for reason of economy. In practice we have typically needed to repeat Steps 1 and 2 three or four times in seeking two-figure accuracy in c (unless $\gamma = c'/c$ is small as in Section II, in which case iteration is unnecessary).

The following examples illustrate the effectiveness of the above procedure for reconstructing c. Both examples are such that the less expensive methods of Sections II and III are not adequate.

EXAMPLE 4. Consider

$$c(\tau) = 1 + 2\sin^2 \pi(\tau - 0.25) \qquad \text{for} \quad 0.25 \leqslant \tau \leqslant 0.75$$
$$= 1 \qquad \text{elsewhere.}$$

Although this c is relatively smooth, it does peak rather sharply near $\tau = 0.5$. Hence it causes difficulty for the approach of Section II since $c' = 4\pi \sin 4\pi(\tau - 0.25)$ and consequently $\gamma = c'/c$ is not "small." This is borne out in computation; e.g., $c_0(0.5) = 2.77$ is off by about 8%. The method of Section III is not appropriate since $c'' = 16\pi^2 \cos 4\pi(\tau - 0.25)$ is too large for the effective use of that approach; see Fig. 3.

If we iterate as outlined above, (19)–(21), we are able to achieve the desired two-figure accuracy by taking $N = 20$ and doing three iterations. Again for efficiency we only correct the first five B_k in Step 1. The results are illustrated in Fig. 3. Computing

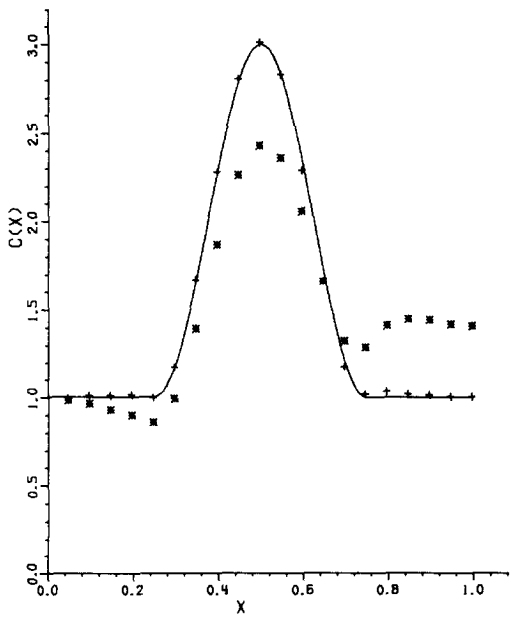

FIG. 3. Graph of $c = 1 + 2\sin^2 2\pi(\tau - 0.25)$ and its approximation due to (19)–(21) denoted by $+$; and by (13)–(15) denoted by $*$. Errors in (19)–(21) solution are: max error = 0.039 and mean error = 0.016.

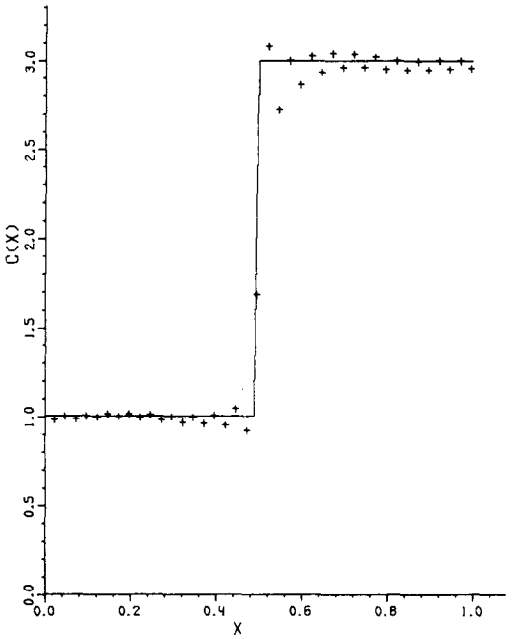

FIG. 4. Graph of step function and its approximation due to four iterations (19)–(21). Final iteration, denoted by +, was not smoothed.

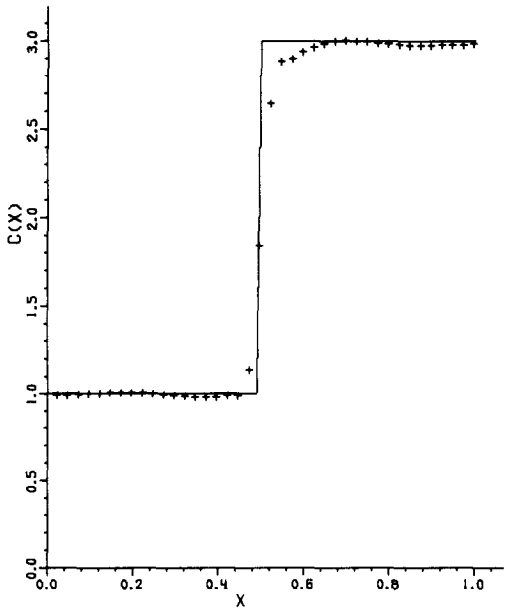

FIG. 5. Graph of step function and its approximation due to four iterations (19)–(21). Final iteration, denoted by +, was smoothed by (22).

time for this example was: 12.7 sec (as compard to about 0.2 sec to get c_0 and 2 sec for the method of Section III when these methods are applicable).

The final example offers a severe test of the robustness of the iteration scheme; c has a large discontinuity, hence c' is a delta function. The Fourier series for c is extremely slow to converge (and the series for γ is divergent; hence the abandoned iteration form (17) is especially unattractive). The proof of convergence of the iteration given in [2] does not apply in this case. Moreover, the Gibbs phenomenon in the series for such a c will eliminate any chance for a good uniform approximation to c. In spite of all this, we shall see that a reasonable reconstruction of c is still possible.

EXAMPLE 5. Consider the step function

$$\begin{aligned} c(\tau) &= 1 \qquad \text{for} \quad \tau < 0.5 \\ &= 3 \qquad \text{for} \quad \tau \geqslant 0.5. \end{aligned}$$

The iteration (19)–(21) was applied with the following simple modification. In solving the differential equations in (20) some consideration should be given to the fact that the "old" c used is actually a truncated Fourier series of an approximation to the step function above. In particular, it will have unwelcome oscillations, especially near $\tau = 0.5$, which will give the differential equation solver some difficulty. We have found that the computer time can be reduced significantly (by over 60% in this example) and somewhat more accurate results can be obtained if the old c is smoothed before it is used in (20) and (19). The following simple smoothing scheme has proved very effective:

$$c(\tau_j) \leftarrow \tfrac{1}{4}[c(\tau_{j-1}) + 2c(\tau_j) + c(\tau_{j+1})]; \qquad j = 2,..., N-1. \tag{22}$$

Figure 4 illustrates the results of taking four iterations with $N = 40$ (but once again we only correct $B_1,..., B_5$). The final c (denoted c_4) was not smoothed in Fig. 4. Smoothing (22) can also be used to make the final c a bit more attractive when graphed; this is shown in Fig. 5. This smoothing can be demonstrated numerically as follows: consider the sub-intervals $[0, 0.4]$ and $[0.6, 1]$ of $[0, 1]$ which exclude the discontinuity at $\tau = 0.5$. On these sub-intervals the average (absolute value) error in the unsmoothed c_4 (of Fig. 4) is 0.025 whereas the average error for the smoothed c_4 (of Fig. 5) is 0.011. In short, smoothing allows us to reconstruct c to two figures (away from the discontinuity). The computer time was 12.0 sec for the results shown in Fig. 4 or Fig. 5; without any smoothing the time was 31.6 sec.

V. SUMMARY

We have discussed three related but significantly different schemes for reconstructing c from the original problem (2) and data $u_s(0; \omega)$. In Section II we

used $u_i = e^{i\omega\tau}$ as our first estimate of u, solution to (2); and demonstrated that if c does not vary much (c'/c small) then c can be reconstructed to perhaps two figures very cheaply (typically 0.1 to 0.2 sec on a fairly slow large scale computer). In Section III we demonstrated that if c is smooth and if c' does not vary too much (c'' moderate) then the WKB estimate $u_i = \sqrt{c}\, e^{i\omega\tau}$ makes an effective first estimate. When applicable, using that approach one can reconstruct c to, typically, two-figure accuracy at modest cost (say 1 to 2 sec) even if c'/c is not small.

Unless the variance of c is quite small, both of the above procedures are unlikely to produce more than two-figure results since they do ignore a term of some significance in integral equation (3). In Section IV we discussed an interative scheme which is more generally applicable. It is considerably more expensive, with two-figure results typically taking 10 to 15 sec (on problems out of the range of the other methods). Moreover, the iteration method is on a sound mathematical foundation as shown in [2], where convergence of the scheme was established. However, achieving, say, six- or eight-figure results would mean overcoming some computational hurdles that we have not dealt with. For example, this would entail solving (20) for large N and thus demand the numerical solution of a large number of highly oscillatory differential equation. We have not pursued these matters since in our applications (and most others) the data are not accurate enough to merit seeking such precision.

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